Covariance Analysis for a Relativity Mission

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An experiment capable of measuring the Lense-Thirring effect of the general theory of relativity has been proposed recently. Satellite-to-satellite Doppler slant range measurements (made over the Earth's poles as the satellites pass) must be processed in order to distinguish the sum of the nonrelativistic orbit plane nodal precessions, which is a function of mean plane separation, from the relativistic nodal precession. A covariance analysis confirms that mean plane separation is determined with sufficient resolution to perform a relativistic measurement to the 1% accuracy previously estimated.[1] Additionally, much improved information is obtained about tesseral harmonic perturbations and about the Earth's tidal elastic response.

Introduction

IN 1974, Van Patten and Everitt[1] proposed a new test of the general theory of relativity. Using general relativity, Lense and Thirring[2] predict that the plane of a body orbiting a rotating mass is dragged a small amount in the direction of rotation of the mass. For any satellite orbit 800 km above the Earth, this relativistic motion is an angular rate of 0.18 arc-sec/yr or a linear rate of 6 m/yr (referred to the Earth's surface).

Other experiments to measure effects of general relativity include the precession of the perihelion of Mercury, the bending of starlight by the sun, and, more recently, time-delay measurements of radar-ranging signals to the planets and interplanetary probes. All of these measure effects of static matter. In 1960, Schiff[3] proposed an experiment, currently under development at Stanford, which measures the effect of the rotating Earth on an orbiting gyroscope. The present experiment applies two counterorbiting satellites to measure the Lense-Thirring nodal drag of the rotating Earth. The idea of applying counterorbiting satellites to measure gravitational effects of rotating matter also had been discussed independently by Miller and Shapiro[4] in 1971 for an experiment about the sun and by Davies[5] in 1973 for an experiment in equatorial orbits about the Earth. A major difficulty in all such experiments is the nonrelativistic perturbation due to the oblateness of the central body. As explained below, the present experiment uses Doppler ranging between the spacecraft to calculate out the oblateness terms with great precision.

The development of the drag-free satellite (which actively compensates for nongravitational forces encountered in orbit), with its inherent orbital stability, paved the way for making this relativistic measurement. At first it was believed that a single drag-free satellite in a polar orbit might be sufficient for making the test. But the single-satellite experiment requires that the satellite orbit plane position history relative to the poles be known to centimeters (!) to reduce the nonrelativistic orbit plane precession uncertainty to a level below that expected by general relativity.

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Two-Satellite Experiment

The two-counterorbiting-satellite experiment has significant advantages. Were it possible to place the two satellites in identical orbits (even if not polar), their planes would precess, because of the nonrelativistic effect of the Earth's oblateness, in opposite directions at the same rate; i.e., the sum of the nonrelativistic nodal precession rates would be zero. It is not possible to place the satellites precisely thus, but now the nonzero sum of the nonrelativistic nodal precession rates is a multiple (related to Earth oblateness and known to four significant figures) of the mean plane difference, which (as has been shown) can be deduced very accurately from satellite-to-satellite Doppler ranging measurements as they pass at or near the poles. Subtracting the mean orbit plane precession due to Earth oblateness (obtained by measuring mean plane separation) from the actual mean orbit plane precession (as measured by ground tracking), a measurement of the relativistic effect is obtained to ~1% accuracy in a 2½-yr experiment provided that the mean orbit plane difference over 2½ yr is less than 15 m, so that the uncertainty in J_2 is unimportant.

Doppler Information

The satellite-to-satellite Doppler ranging measurement picks up substantially more than just a mean plane separation. Direct solar and lunar gravity gradients produce periodic lateral separation fluctuations at the poles of approximately 1100 m and 180 m, respectively. Tides raised on

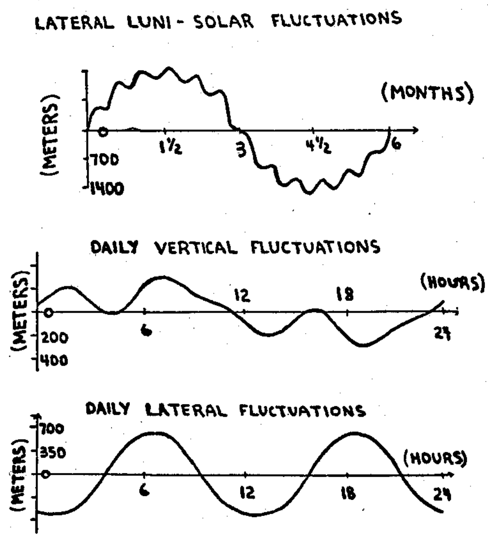


Fig. 1 Dominant fluctuations at the poles.

the Earth's surface add an additional ~15% to this. (Large secular effects of the solar and lunar gravity gradients are eliminated by locating the orbit planes near the pole of the ecliptic.) Tesseral harmonics of the Earth's gravity field (J_{lm}) with even m produce lateral fluctuations m times per day at the poles, with amplitudes as large as 550 m. Tesseral harmonics with m odd produce vertical fluctuations m times per day at the poles. Harmonics like J_{41} with even l and odd m have a relatively minor effect on the vertical separation and introduce a small difference between the daily histories of the vertical separations at the two poles. Harmonics like J_{32} with odd l and even m have a similar small effect on the lateral separations. Therefore, m times per day fluctuations (both lateral and vertical) at the north pole are different from m times per day fluctuations at the south pole (Fig. 1). The direct effect of the lunar gravity gradient produces monthly relative altitude fluctuations of about 10 m. Slight orbital eccentricity differences also will produce altitude fluctuations. Graziani et al[6] showed that longitude differences in the elastic tidal response of the Earth will appear as lateral fluctuations at integral numbers of times per day, modulated by solar and lunar periods (i.e., what appear to be seasonal fluctuations in tesseral harmonics). North-south elastic asymmetry of the Earth will produce twice-yearly and twice-monthly vertical fluctuations. These effects have been included in a covariance analysis of six months of Doppler data at a single pole.

Since Doppler measurement accuracies over the maximum 2 km polar separation approach 1 cm, a good model must include all parameters that produce perturbations down to at least this level. Theoretical estimates of the magnitude of J_{lm} are given by Allan:

$$|\bar{J}_{lm}| \sim \frac{\sqrt{12.2\times 10^{-10}}}{(2l+1)\sqrt{2l+3}}\,(0.93)^{l+(3/2)}$$

Based on this, the estimated effect of a particular even m due to all even l on the orbital inclinations of the satellites over the poles is

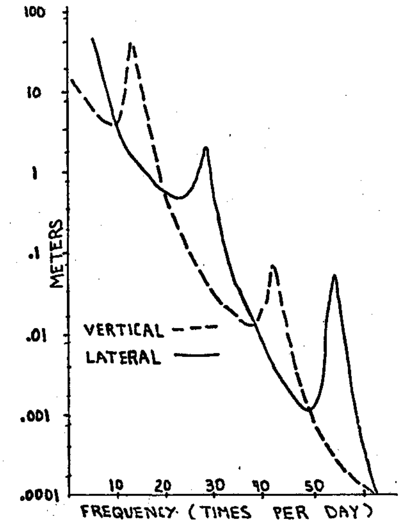
$$|\delta i(\frac{\pi}{2})\text{even} \sim \frac{3.1\times 10^{-5}[0.93(R_\oplus/a)]^m}{\pi^{1/4}m^{1/4}\{1-[0.93(R_\oplus/a)]^4\}^{1/4}} \times\left|\frac{1}{m\omega_\oplus/n}+2\sum_{k=1}^{\infty}\frac{(-1)^k(m\omega_\oplus/n)}{(m\omega_\oplus/n)^2-4k^2}\right|$$

The identical formula gives the estimated effect of a particular odd m due to all odd l on the altitude fluctuations of the satellites over the poles. The estimated effect of a particular even m due to all odd l on the orbital inclination of the satellites over the poles is

$$|\delta i(\frac{\pi}{2})|\text{odd} \sim \frac{8.8\times 10^{-5}[0.93(R_\oplus/a)]^{m+1}}{\pi^{1/4}m^{5/4}\{1-[0.93(R_\oplus/a)]^4\}^{1/4}} \times\left|\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)(m\omega_\oplus/n)}{(m\omega_\oplus/n)^2-(2k+1)^2}\right|$$

Correspondingly, the identical formula gives the estimated effect of a particular odd m due to all even l on the altitude fluctuations of the satellites over the poles. $R_\oplus$ is the radius of the earth, a is the satellite's semimajor axis, n is the mean orbital rate, $\omega_\oplus$ is the Earth's rotation rate, and l and m are the indices of J_{lm} (see Fig. 2). As a result of these estimates, the model includes the magnitudes and phases of tesseral harmonics, m even through $m=60$ and m odd through $m=59$. Solar (☉) and lunar (☾) gravity gradients and tidal effects were included in the lateral direction as well as the gradient of the lunar gravity gradient in the vertical direction. Frequencies $m\omega_\oplus \pm 2$☾ and $m\omega_\oplus \pm 2$☉, corresponding to seasonally varying tesseral coefficients, were included in the lateral direction through $m=8$. Two plane separations and two eccentricities also were included. This model results in a total of 202 parameters.

Fig. 2 Estimated amplitude of fluctuations over poles due to $C_{\cdot m}$.



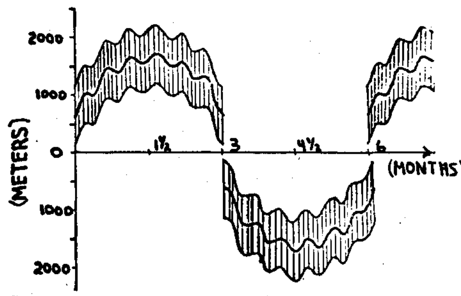
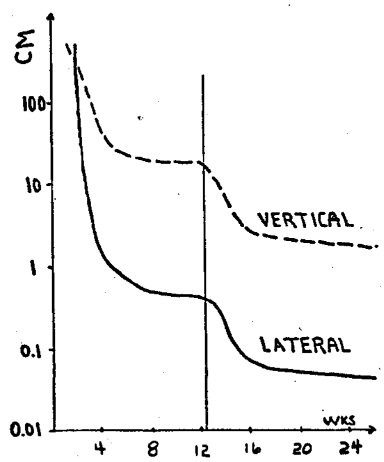


Fig. 3 Lateral separation history with 3-month plane corrections.

Fig. 4 Covariance evolution of tesseral harmonic parameters.



The reason two of each of the latter parameters are required is that, because of a satellite collision possibility,[7] it was deemed necessary to alter the orbit planes slightly every three months (Fig. 3). Applying a small crosstrack impulse to one or both of the orbit planes as the satellites cross the equator effects a collision avoidance maneuver without disturbing the position of the nodes and enables the mean plane separation to be held below 15 m over the 2½-yr period.

Because of the difference between the fluctuation models at the north and south poles, it was sufficient to analyze information from a single pole. Information was accumulated once per orbit over a representative 6-month period, giving a total of 1300 measurements. A 6-month period is "representative" inasmuch as it includes at least one full cycle of all the major fluctuations present. Extrapolation to the

2½-yr experiment with information from both poles is performed using simple statistical laws, which is slightly pessimistic.

Since the Doppler signal effectively measures minimum range, which is a nonlinear function of vertical and lateral separations, linearization about a "nominal" separation was necessary (see Appendix A). It was found that the covariances obtained were very insensitive to different recent "nominal" descriptions of the Earth's gravity field. The variations were, at worst, ~4%.

The accuracy of Doppler determination of minimum range was assumed to vary with range to be measured. Anderson and Breakwell[8] give the theoretical Doppler ranging accuracy as

$$\frac{\sigma_r}{r} = \left(\frac{8}{\pi}\right)^{1/2} \frac{c}{V} \frac{\sigma_f}{f} \left(\frac{Vt_c}{r}\right)^{1/2}$$

where r is the distance of closest approach, V is the relative velocity, f is the nominal frequency, σ_r and σ_f are the standard deviations of r and f, and t_c is the correlation time for σ_f.

Using state-of-the-art crystal oscillator electronics[9] (i.e., $\sigma_f/f \sim 10^{-9}$ with $t_c = 0.1$ sec), the range error actually should approach the theoretical Doppler limit of 1 cm. The Doppler oscillator frequency can be estimated, together with the minimum range distance at each encounter, thus eliminating frequency biases. The oscillator frequency over the brief radio encounter period (~1 sec) is very stable, minimizing the effects of frequency drift. In the orbital environment, the refractive effects of the atmosphere are eliminated, and, because of the proximity of the spacecraft at the encounter, the effects of ionized particles in the radio path will be negligible. Since the range measurement is made relative to the satellites, no ground station errors are introduced. With corrections for spacecraft attitude (to 1°) and the location of the Doppler antenna with respect to the proof mass, the minimum distance between the proof masses can indeed be determined to ~1 cm.

This model processes polar slant range information as if the minimum satellite-to-satellite slant range occurs directly over the poles. In reality, this encounter may miss the poles by as much as 2 km in-track and 2 km crosstrack. Additionally, the orbit planes will "scissor" slightly (i.e., ascending nodes not remaining exactly 180° apart), producing a different encounter geometry. Therefore, the off-polar data must be "corrected to the pole" before using this model. Since the satellites miss the pole, they pick up small components of the tesseral harmonics which produce fluctuations at the equator. The effects of the fluctuations at the equator are very much attenuated (by a factor 2 km/a=0.00028) by the time the satellites reach the pole, and so it is sufficient to use smoothed ground tracking (accurate to ~10 m and already required to measure the position of the node for the relativity experiment) to obtain the equatorial fluctuations. The components of the equatorial fluctuations which appear in the data near the pole

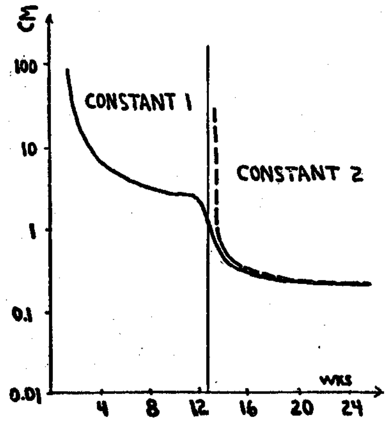


Fig. 5 Covariance evolution of mean plane separation parameters.

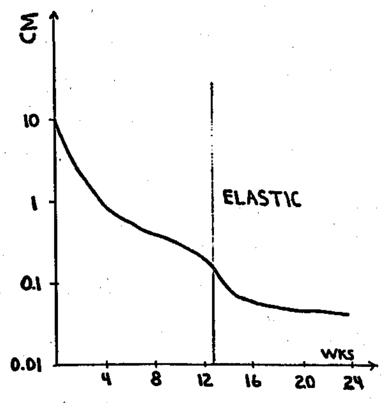


Fig. 6 Covariance evolution of elastic Earth parameters.

then can be subtracted from the slant range measurement. A geometrical correction also is needed to account for the satellites' velocities not being exactly parallel to each other (scissoring) and not being perpendicular to the polar axis of the Earth at closest approach.

For a maximum allowable orbit plane scissor angle Ω of 1 km/a and an in-track polar miss angle θ^* of 2 km/a, the correction to the pole of the measured minimum range is $a\Omega\theta^*$. The scissor angle Ω is extremely well known, and the ephemerides of the drag-free satellites determine θ^* to 10 m in-track accuracy. The error introduced into the determination of the minimum range at the pole is thus negligible.

Covariance Analysis

A computer program employing a Householder transformation technique (Appendix B) was used to sequentially update the information matrix (i.e., the inverse of the covariance matrix). This technique is preferable to a Kalman filter update procedure for its superior numerical accuracy. The update procedure was started with a pessimistic initial information matrix (although a priori information is not required for starting) whose diagonal elements were taken to be substantially less than the diagonal elements of the information matrix after 6 months.

The total of 1300 measurements was processed in batches of 220 measurements or ~1/24 yr. The data could have been processed slightly more efficiently in larger batches; however, additional information was obtained by studying the evolution of the diagonal elements of the covariance matrix at intervals of 1/24 yr.

Conclusions

Typically, the following results were obtained for a 6-month experiment:

1) All periodic lateral fluctuations (m even) are measurable to accuracies of 0.5 mm (Fig. 4).

2) All periodic vertical fluctuations (m odd) are measurable to accuracies of 2.5 cm (Fig. 4).

3) Mean plane separations are measurable to an accuracy of 0.2 cm. Converting this into its corresponding nonrelativistic nodal precession uncertainty for the full 2½-yr experiment gives 0.05 m, as compared with 30 m due to the Lense-Thirring effect (Fig. 5).

4) K_2, the Love number associated with the Earth's elastic response to a second harmonic disturbing function, will be measurable to ~1 part/10^5. It is presently known to a few parts in 10^3. K_3, the Love number associated with the Earth's elastic response to a third harmonic disturbing function, will be measurable to 0.001. No determination of K_3 has been made at present (Fig. 6).

The results of this covariance analysis point out several significant facts. First, the covariances obtained are virtually insensitive to the gravity model used in computing the nominal lateral and vertical fluctuations. The covariances are also insensitive to the relative phases of the sun and moon at the start of the experiment.

Second, the problem that was solved is well conditioned, in spite of having to distinguish several close frequencies (e.g., tesseral harmonics and seasonally modulated tesseral harmonics), and in spite of having to distinguish twice per year lateral fluctuations from twice per year vertical fluctuations (which is accomplished only through the variable geometry during the dominant, twice yearly, lateral fluctuation) (see Appendix A).

Third, orbital resonances with the tesseral harmonics must be avoided so that pseudosecular amplification of these harmonics does not occur (permitting data to be accumulated at all times of the sidereal day). For example, an orbital rate of 14.25 orbits/day (corresponding to an 800-km orbital altitude) is not resonant with any low-order tesseral but is resonant with tesseral harmonics of order 57. Since the magnitudes of these tesserals are probably quite small, even the resonance will not produce large fluctuations. However, with this orbital frequency, the timing of our measurement would make it impossible to distinguish between 56 and 58 times/day fluctuations.

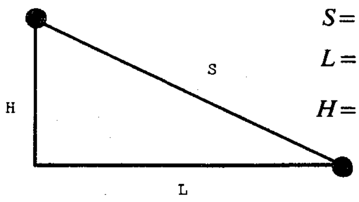
Fourth, parameters producing fluctuations in the lateral direction are determined much more accurately than similar vertical fluctuations. The reason for this is that the Doppler range measurement is mainly a lateral measurement (because the nominal relative lateral excursions of the satellites are much larger than the vertical excursions). Consequently, there is a much greater sensitivity to fluctuations in the lateral direction.

An additional feature presented in Figs. 4-6 is the sudden drop in the covariances at the 3-month mark. At this time, the satellites are at their closest approach in the lateral direction. The accuracy in the model is improved for two reasons. The satellites, at minimum lateral separation, are at the maximum sensitivity for determining vertical fluctuations. Furthermore, at this minimum range configuration, the Doppler measurement accuracy is best. These two effects combine to produce a drop in all of the covariances observed after 3 months.

Once a collision avoidance maneuver is performed, there is no way to directly accumulate more information on the old mean plane separation and eccentricity. However, our filter has a "memory," and so, as more and more data are added, our entire model is improved, and hence a higher accuracy can be assigned to a parameter that was measured directly only during previous periods. This accounts for the further decrease in the covariance of the mean plane separation constant in Fig. 5 after the 3-month mark.

As a result of this covariance analysis, the satellite collision problem can be viewed from a somewhat different perspective. Mean plane separation can be determined to an accuracy that reduces the uncertainty in the nonrelativistic node precession from a major error source of the relativity experiment to a fairly minor source. With this knowledge, it is possible to introduce a relative orbital eccentricity between the satellites at launch in such a way as to avoid a collision, with certainty, for as long as 5 months. Crosstrack measurement accuracy is degraded slightly, but as was mentioned previously, this is no longer crucial. Measurements in the vertical direction are now improved. After accumulating satellite-to-satellite data for 5 months and producing a fluctuation model good to centimeters, it will be possible to predict in advance if and when a collision will occur. Then, only if a collision threatens (a probability estimated to be about 10%) will a maneuver be required. This offers the possibility of performing the entire 2½-yr experiment without maneuvering. In conclusion, the prospects for performing the measurement of the Lense-Thirring effect of the general theory of relativity are very good and offer, in addition, a chance of obtaining geophysical information of high accuracy.

Appendix A: Linearization of Doppler Slant Range Measurement About Nominal



$$S=\sqrt{L^2+H^2}$$
$$L=\sum (A_i\cos\omega_i t+B_i\sin\omega_i t)$$
$$H=\sum (C_j\cos\omega_j t+D_j\sin\omega_j t)$$

where A_i, B_i, C_j, D_j are the parameters to be determined.

Expanding S about S_{nom}

$$S=S_{nom}+(L_{nom}/S_{nom})\sum(\delta A_i\cos\omega_i t+\delta B_i\sin\omega_i t)$$
$$+(H_{nom}/S_{nom})\sum(\delta C_j\cos\omega_j t+\delta D_j\sin\omega_j t)$$

Making various slant range measurements at various times gives a linear least-squares problem for the parameters A_i, B_i, C_j, D_j.

The "nominals" are computed using the dominant terms in the trigonometric expansions. In this analysis, L_{nom} included the solar and lunar gravity gradients, the mean plane separation, tesseral harmonics through J_{44}; and H_{nom} included tesseral harmonics through J_{55}.

Appendix B: Householder Transformation

Consider the least-squares solution to the linear problem $Ax=b$:

$$A\ m\times n,\ x\ n\times 1,\ b\ m\times 1$$

If the matrix T is orthogonal, we may solve the equivalent problem $TAx=Tb$. Furthermore, if T has the property that

$$TA=\begin{matrix} \\ n \\ \\ m\end{matrix}\begin{bmatrix} \ddots & U \\ 0 & \ddots \\ \hline & 0 \end{bmatrix}^{\,n} \qquad T_b=\begin{bmatrix} C \\ \hline e \end{bmatrix}$$

then the minimizing solution to $Ax=b$ is given by the back substitution problem $Ux=C$, and, furthermore,

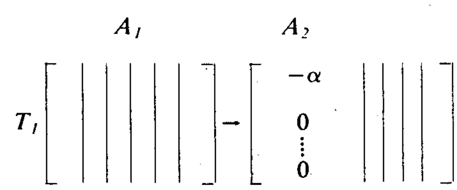
$$\mathrm{cov}(\tilde{x})=(A^TA)^{-1}=(A^TT^TTA)^{-1}=(U^TU)^{-1}$$

where $\tilde{x}=x-\hat{x}$.

Consider the vector

$$a^T=(a_1,\ldots,a_K,\ldots,a_m)$$
$$\alpha=\mathrm{sgn}(a_K)\,[a_K^2+\ldots a_m^2]$$
$$u^T=[0,0,\ldots,a_K+\alpha,\ldots,a_m]$$
$$\beta=\alpha u_K=\alpha a_K+\alpha^2$$

Then $T=I-1/\beta\ uu^T=I-2\hat{u}\hat{u}^T$ is an orthogonal transformation (known as a Householder reflection), leaving $a_1,\ldots,a_{K-1}$ fixed, changing a_K to $-\alpha$, and zeroing $a_{K+1},\ldots,a_m$. So



$$\begin{matrix} & A_2 & & & A_3 & & \\ T_2 & \begin{bmatrix} -\alpha_1 & | & | & | & | \\ 0 & | & | & | & | \\ \vdots & | & | & | & | \end{bmatrix} & \rightarrow & \begin{bmatrix} -\alpha_1 & -- & | & | & | & | \\ 0 & -\alpha_2 & | & | & | & | \\ \vdots & 0 & | & | & | & | \\ 0 & 0 & | & | & | & | \end{bmatrix} \end{matrix}$$

etc. Since the product $T = T_n T_{n-1} \ldots T_2 T_1$ is also orthogonal, T is the required transformation.

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